# ON THE FOURIER ANALYTIC STRUCTURE
# OF THE BROWNIAN GRAPH

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ABSTRACT. In a previous article (*Int. Math. Res. Not.* 2014, 2730–2745) T. Orponen and the authors proved that the Fourier dimension of the graph of any real-valued function on $\mathbb{R}$ is bounded above by 1. This partially answered a question of Kahane ('93) by showing that the graph of the Wiener process $W_t$ (Brownian motion) is almost surely not a Salem set. In this article we complement this result by showing that the Fourier dimension of the graph of $W_t$ is almost surely 1. In the proof we introduce a new method based on Itō calculus to estimate Fourier transforms by reformulating the question in the language of Itō drift-diffusion processes and combine it with the classical work of Kahane on Brownian images. We also give some applications to the equidistribution of orbits under toral endomorphisms.

## 1. INTRODUCTION AND RESULTS

1.1. **Geometric properties of Brownian motion.** Gaussian processes are standard models in modern probability theory and perhaps the most well studied example is the *Wiener process* (or standard *Brownian motion*) $W = W_t$ defined such that $W_0 = 0$, the map $t \mapsto W_t$ is almost surely continuous and $W_t$ has independent increments such that $W_t - W_s$ for $t > s$ is normally distributed:

$$W_t - W_s \sim N(0, t - s).$$

The Wiener process has far reaching importance throughout mathematics and it is a topic of particular interest to understand the geometric structure of the process. This can be achieved by studying several random fractals associated to the process such as *images* $W(K) := \{W_t : t \in K\}$ of compact sets $K \subset [0, \infty)$, *level sets* $L_c(W) := \{t \in \mathbb{R} : W_t = c\}$ for $c \in \mathbb{R}$, *graphs* $G(W) := \{(t, W_t) : t \in \mathbb{R}\}$ and other more delicate constructions such as $\mathrm{SLE}_\kappa$-curves.

Due to the properties of the Brownian motion, these random fractals enjoy a certain statistical self-similarity which facilitates computation of their Hausdorff dimensions $\dim_H$. Classical results include McKean's proof [27] that $\dim_H W(K) = \min\{1, 2\dim_H K\}$ almost surely for each compact $K \subset [0, \infty)$. Moreover, for the level sets $\dim_H L_c(W) = 1/2$ almost surely for $c = 0$ by Taylor [37] and for all $c \in \mathbb{R}$ by Perkins [31] whenever $L_c(W)$ is non-degenerate. For the Brownian graph $G(W)$, Taylor [36] proved that $\dim_H G(W) = 3/2$ almost surely and Beffara computed Hausdorff dimensions of $\mathrm{SLE}_\kappa$-curves [3]. Moreover, Hausdorff dimensions for similar sets given by many other Gaussian processes, such as fractional Brownian motion,

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have been also considered, see for example Adler's classical results [1] for fractional Brownian graphs and the recent work for variable drift by Peres and Sousi [30].

1.2. **Fourier analytic properties of Brownian motion.** The Hausdorff dimension is the most commonly used tool for measuring the size of a set $A$ but there is also another fundamental notion based on Fourier analysis which reveals more arithmetic and geometric features of $A$, including curvature, which are not seen by the Hausdorff dimension. This is based on studying the *Fourier coefficients* of a probability measure $\mu$ on $A \subset \mathbb{R}^d$, which are defined by

$$\widehat{\mu}(\xi) := \int e^{-2\pi i \xi \cdot x} \, d\mu(x), \quad \xi \in \mathbb{R}^d.$$

Now the size of $A$ can be linked to the existence of probability measures $\mu$ on $A$ with decay of Fourier coefficients $\widehat{\mu}(\xi)$ when $|\xi| \to \infty$. The following connection between Hausdorff dimension and decay of Fourier coefficients is well-known and goes back to Salem and Kaufman, but we refer the reader to [25] for the details. If $\dim_{\mathrm{H}} A > s$, then $A$ supports a probability measure $\mu$ with $|\widehat{\mu}(\xi)| = O(|\xi|^{-s/2})$ "on average", that is, $\int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} \, d\xi < \infty$ and vice versa the Hausdorff dimension can be bounded from below if such a measure $\mu$ can be found. It is possible, however, that $\dim_{\mathrm{H}} A = s > 0$ but no measure $\mu$ on $A$ has Fourier decay at infinity, this happens for example when $A$ is the middle-third Cantor set on $\mathbb{R}$. Therefore, one defines the notion of *Fourier dimension* $\dim_{\mathrm{F}} A$ of a set $A \subset \mathbb{R}^d$ as the supremum of $s \in [0, d]$ for which there exists a probability measure $\mu$ supported on $A$ such that

$$|\widehat{\mu}(\xi)| = O(|\xi|^{-s/2}), \quad \text{as } |\xi| \to \infty. \tag{1.1}$$

Then by this definition we always have $\dim_{\mathrm{F}} A \leqslant \dim_{\mathrm{H}} A$ and if the two dimensions coincide then $A$ is called a *Salem set* or a *round set* after Kahane [20]. In general Fourier dimension and Hausdorff dimension have no relationship other than this; in fact, Körner [11] established that for any $0 \leqslant s < t \leqslant 1$ it is possible to construct examples $A \subset \mathbb{R}$ with $\dim_{\mathrm{F}} A = s$ and $\dim_{\mathrm{H}} A = t$. Further theory for Fourier dimension was recently developed by Ekström, Persson and Schmeling [8]. For a more in depth account of Fourier dimension, the reader is referred to [24, 25].

Finding measures $\mu$ on $A$ with polynomially decaying Fourier transform (i.e. (1.1) for some $s > 0$) has deep links to absolute continuity, arithmetic and geometric structure and curvature. If $A$ supports a measure $\mu$ such that (1.1) holds with $s > 1$, then Parseval's identity yields that $\mu$ is absolutely continuous to Lebesgue measure and $A$ must contain an interval. An application of Weyl's criterion known as the Davenport-Erdös-LeVeque criterion (see [7]) yields that in $\mathbb{R}$ polynomial decay of $\widehat{\mu}$ guarantees that $\mu$ almost every number is normal in every base and a interesting result of Łaba and Pramanik [13] shows that if the $s$ in (1.1) is sufficiently close to 1 for a measure $\mu$ on $A \subset \mathbb{R}$, then $A$ contains non-trivial 3-term arithmetic progressions. Moreover, an analogous result also holds for higher dimensions with arithmetic patches [6]. On the curvature side, if $A$ is a line-segment in $\mathbb{R}^2$, then $A$ cannot contain any measure with Fourier decay at infinity so $A$ cannot be a Salem set. However, if $A$ is an arc of a circle or more generally a 1-dimensional smooth manifold with non-vanishing curvature when the 1-dimensional Hausdorff measure $\mu$ on $A$ satisfies (1.1) with $s = 1$, see [25]. In particular, $A$ is a Salem set. In these examples of $A$ one can observe that the important arithmetic or curvature features present are not seen from the Hausdorff dimension.

Constructing explicit Salem sets (which are not manifolds), or just sets $A$ supporting a measure $\mu$ satisfying (1.1) for some $s > 0$, can be achieved through, for example, Diophantine approximation by Kaufman's works [22, 23], Bluhm [4] or via thermodynamical tools by Jordan and Sahlsten [12]. However, for random sets it has been observed in many instances that $A$ is either almost surely Salem or at least supports a measure $\mu$ with (1.1) for some $s > 0$. This was first done for random Cantor sets by Salem [34], where Salem sets were also introduced. Later Kahane published his classical papers [16, 17], where he found out that the Wiener process and other Gaussian processes provide natural examples.

The study of Fourier analytic properties of natural sets for Gaussian processes and more general random fields has since been an active topic. For the Brownian images Kahane [19] proved that for any compact $K \subset \mathbb{R}$ the image $W(K)$ is almost surely a Salem set of Hausdorff dimension $\min\{1, 2 \dim K\}$. Kahane also established a similar result for fractional Brownian motion. Łaba and Pramanik [13] then applied these to the additive structure of Brownian images. Later Shieh and Xiao [35] extended Kahane's work to very general classes of Gaussian random fields. However, understanding the Fourier analytic properties of the level sets and graphs remained an important problem for some time. In 1993, Kahane [20] outlined the problem explicitly.

**Problem 1.1** (Kahane). *Are the graph and level sets of a stochastic process such as fractional Brownian motion Salem sets?*

In this form Kahane's problem was formulated by Shieh and Xiao [35, Question 2.15]. For the Wiener process Kahane [18] had already established that the level sets $L_c(W)$ are Salem almost surely for any fixed $c \in \mathbb{R}$ provided that $L_c(W)$ is non-degenerate. The fractional Brownian motion case has recently been considered in the case $c = 0$ by Fouché and Mukeru [9].
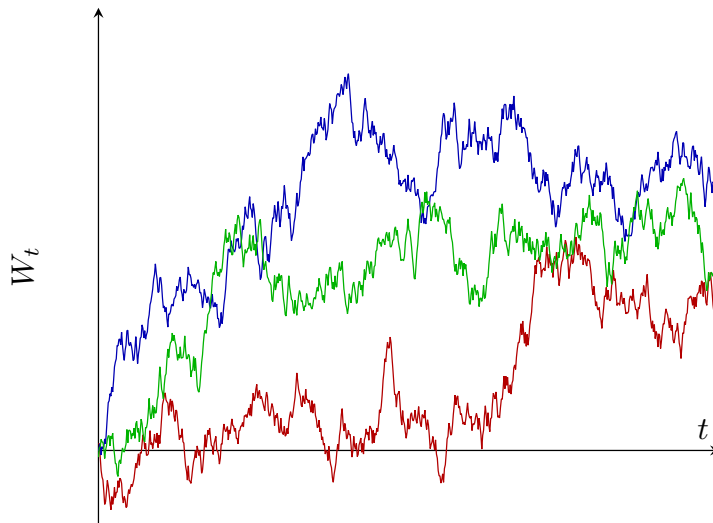


FIGURE 1. Three realisations of the graph $G(W)$ for the Brownian motion $W_t$.

Kahane's problem for graphs, even in the case of the standard Brownian motion $W_t$, however, remained open for quite a while until, together with T. Orponen, we established that the

Brownian graph $G(W)$ is almost surely *not* a Salem set [14]. It turned out that the reason for this is purely geometric: the proof was based on the following application of a Fourier-analytic version of Marstrand's slicing lemma.

**Theorem 1.2** (Theorem 1.2 in [14]). *For any function $f : [0, 1] \to \mathbb{R}$ the Fourier dimension of the graph $G(f)$ cannot exceed 1.*

Indeed, since the Hausdorff dimension $\dim_{\mathrm{H}} G(W) = 3/2 > 1$ almost surely (see [36]), this result shows that the graph $G(W)$ cannot be a Salem set almost surely, which answers Problem 1.1 in the negative for the Wiener process. Note that this also gives a negative answer for fractional Brownian motion since the Hausdorff dimension in that case is also strictly larger than 1 almost surely.

The methods in [14] are purely geometric and involve no stochastics, so they do not shed any light on the precise value for the Fourier dimension of $G(W)$. Note that even though $\dim_{\mathrm{H}} G(f) \geqslant 1$ for any continuous $f : [0, 1] \to \mathbb{R}$ by Theorem 1.2, it is completely possible that the Fourier dimension $\dim_{\mathrm{F}} G(f) = 0$. For example this happens by just taking $f$ to be affine or even any *Baire generic* $f \in C[0, 1]$ satisfies $\dim_{\mathrm{F}} G(f) = 0$, see [14, Theorem 1.3].

The main result of this paper is to complete the work initiated by Kahane's Problem 1.1 in the case of Brownian motion by establishing the precise almost sure value of the Fourier dimension of $G(W)$.

**Theorem 1.3.** *The graph $G(W)$ has Fourier dimension $1$ almost surely.*

Moreover, the random measure $\mu$ we use to realise the Fourier dimension is Lebesgue measure $dt$ on $[0, 1]$ lifted onto the graph $G(W)$ via the mapping $t \mapsto (t, W_t)$. The precise estimate we obtain is that almost surely

$$|\widehat{\mu}(\xi)| = O(|\xi|^{-\frac{1}{2}} \sqrt{\log |\xi|}), \quad \text{as } |\xi| \to \infty, \tag{1.2}$$

which combined with Theorem 1.2 yields Theorem 1.3.

A natural direction in which to continue this line of research would be to study other Gaussian processes with different covariance structure, see the discussion in Section 5.1.

1.3. **Methods: Itō calculus and reduction to Brownian images.** The key method we introduce to estimate the Fourier transform of the graph measure $\mu$ is based on *Itō calculus*, which has previously been a natural framework in the theory of stochastic differential equations. As far as we know, Itō calculus has not been previously considered in this Fourier analytic context. In Section 5 we will discuss the prospects of this approach. Next we will discuss this method and give a brief summary of the main steps in the proof. When written in polar coordinates, (1.2) asks about the rate of decay for the integral

$$\widehat{\mu}(\xi) = \int_0^1 \exp(-2\pi i u(t \cos\theta + W_t \sin\theta)) \, dt$$

for $\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2$, $u > 0$, $\theta \in [0, 2\pi)$, as $u \to \infty$. There are two distinct cases we will consider depending on the direction of $\xi$, which we give a heuristic description of here.

If we ignore the random component $W_t \sin\theta$, that is, set $\theta = 0$ or $\pi$, then standard integration using the chain rule shows that $\widehat{\mu}(\xi)$ equals the Fourier transform of Lebesgue measure $dt$ at $u$, which decays to 0 with the polynomial rate $u^{-1} = |\xi|^{-1}$, so we are done for these directions.

However, if $\theta$ is not equal to $0$ or $\pi$, we still have a small random (non-smooth) term $W_t \sin\theta$, so a classical change of variable formula or other tools from classical analysis cannot be used.

The new approach we introduce is to utilise tools from stochastic analysis. The key observation is that we can write $\widehat{\mu}(\xi) = \int \exp(iX_t)\, dt$, where the stochastic process

$$X_t := bt + \sigma W_t$$

is a so called *Itō drift-diffusion process*, where $b := -2\pi u \cos\theta$ is the *drift coefficient* of $X_t$ and $\sigma := -2\pi u \sin\theta$ is the *diffusion coefficient* of $X_t$. Itō processes have many useful analytic tools from Itō calculus (see Section 2) associated to them, in particular *Itō's lemma*, which works as an analogue for the chain rule. The price we pay is that Itō's lemma introduces some multiplicative error terms involving stochastic integrals, but they can be estimated with other tools from Itō calculus using moment analysis.

The estimates we obtain from Itō calculus allow us to obtain the correct Fourier decay (1.2) for $\mu$ when $\theta$ is close to $0$ or $\pi$ with respect to $u^{-1}$ (more precisely, $|\sin\theta| < u^{-1/2}$). In other words, when $\xi$ is close to pointing in the horizontal directions. Thus another estimate is needed for $\theta$ bounded away from $0$ and $\pi$. This is where Kahane's classical work [19] on Brownian images comes into play. If we completely ignore the deterministic component $t \cos\theta$, by setting $\theta = \pi/2$ or $3\pi/2$, then $\widehat{\mu}(\xi)$ is the Fourier transform the Brownian image measure $\nu$, that is the $t \mapsto W_t$ push-forward of the Lebesgue measure $dt$ on $[0,1]$ at $u$. Kahane [19] in fact already established that almost surely the decay of $|\widehat{\nu}(u)|$ is almost surely of the order $u^{-1}\sqrt{\log u} = |\xi|^{-1}\sqrt{\log|\xi|}$ so (1.2) holds for these directions. Now a simple modification to Kahane's argument reveals that whenever $\theta \neq 0$ or $\pi$, then almost surely

$$|\widehat{\mu}(\xi)| = O(|\sin\theta|^{-1}|\xi|^{-1}\sqrt{\log|\xi|})$$

see the discussion in Section 3.3. Now one notices that when $\theta$ approaches $0$ or $\pi$, this estimate blows up, and so one cannot obtain a uniform estimate over all directions from this. However, this gives (1.2) if $|\sin\theta| \geqslant u^{-1/2}$, so combining with the estimates we obtained through Itō calculus, we are done. See Section 3 for more details on the main steps of the proof.

1.4. **Equidistribution.** As mentioned in the introduction, Fourier coefficients of $\mu$ on a set $A$ are deeply linked to arithmetic properties of the set $A$. In this section we will point out one application one can achieve from Theorem 1.3 (in particular the decay (1.2)). Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ be the unit circle or 1-torus, which is naturally identified with $[0,1]$. A sequence $(x_n) \subset \mathbb{R}$ is *equidistributed* if

$$\frac{\sharp\{1 \leqslant n \leqslant N : x_n \in [a,b]\}}{N} \to b - a$$

as $N \to \infty$ for any interval $[a,b] \subset [0,1]$. If the sequence $x_n = p^n x \mod 1$ equidistributes for a fixed $p \in \mathbb{N}$, then we say $x$ is *normal* in base $p$. Checking normality of a specific $x$ is often difficult so instead one often tries to check if a *random* number (with respect to a measure $\mu$) is normal. This is where the decay of the Fourier coefficients of $\mu$ are crucial:

**Theorem 1.4** (Davenport-Erdös-LeVeque). *Let $\mu$ be a measure on $\mathbb{R}$ for which there is some $\alpha > 0$ such that*

$$|\widehat{\mu}(\xi)| = O(|\xi|^{-\alpha}), \quad as \ |\xi| \to \infty.$$

*Then for any strictly increasing sequence of reals $(s_n)_{n\in\mathbb{N}}$ the sequence $(s_n x \mod 1)$ is equidistributed for $\mu$ almost every $x$. In particular, $\mu$ almost every $x$ is normal in every base.*

Davenport-Erdös-LeVeque criterion was established in [7] and in this form by Queffélec-Ramaré [33] and it is a consequence of the classical *Weyl's equidistribution criterion* (1908), which says that a sequence $(x_n) \subset \mathbb{T}$ equidistributes if and only if the exponential sums

$$\left| \sum_{n=1}^{N} \exp(2\pi i k x_n) \right| = o(N), \quad \text{as } N \to \infty,$$

for any $k \in \mathbb{Z} \setminus \{0\}$. In higher dimensions one can also study analogues of equidistribution on the $d$-torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Here a sequence $(x_n) \subset \mathbb{T}^d$ is *equidistributed* if in the weak topology

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{x_n} = \text{Haar measure on } \mathbb{T}^d.$$

Weyl's criterion in the higher dimensions has the following form:

**Theorem 1.5** (Weyl's equidistribution criterion)**.** *A sequence $(x_n) \subset \mathbb{T}^d$ is equidistributed if and only if for any $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i\, \mathbf{k} \cdot x_n) = 0.$$

The natural analogue of sequences $(x_n)$ on the torus is to consider orbits under suitable expanding maps. For example, being normal is base $p$ precisely means that for the $\times p$ map

$$T_p(x) := px \mod 1$$

the orbit $\{T_p^n x\}_{n\in\mathbb{N}}$ of $x$ is equidistributed. The map $T_p$ is an example of an *expanding toral endomorphism*, that is, $p^n x$ grows in norm as $n \to \infty$ and $T_p$ preserves the algebraic structure of the unit circle $\mathbb{T}$. A similar notion can be defined for *toral endomorphisms* on $\mathbb{T}^d$, that is, maps of the form

$$T(x) = Ax \mod 1$$

for some matrix $A \in \mathrm{GL}_d(\mathbb{Z})$. We say that the map $T$ is *expanding* if the singular values of $A$ are all strictly greater than 1. We prove the following higher dimensional analogue of the Davenport-Erdös-LeVeque criterion:

**Theorem 1.6.** *Let $\mu$ be a measure on $\mathbb{R}^d$ for which there is some $\alpha > 0$ such that*

$$|\widehat{\mu}(\xi)| = O(|\xi|^{-\alpha}), \quad \text{as } |\xi| \to \infty.$$

*Then the $T$-orbit $\{T^n(x)\}_{n\in\mathbb{N}}$ of $x$ equidistributes at $\mu$ almost every $x$ for any expanding toral endomorphism $T : \mathbb{T}^d \to \mathbb{T}^d$.*

The proof of Theorem 1.6 follows the same lines as the $\mathbb{R}$-version by Davenport-Erdös-LeVeque and we postpone the proof until Section 4. Thus as an immediate corollary of Theorem 1.3, we obtain the following equidistribution theorem for Brownian motion:

**Corollary 1.7.** *Almost surely for Lebesgue almost every $t \in [0,1]$ the orbit $\{T^n(t, W_t)\}_{n\in\mathbb{N}}$ equidistributes for any expanding toral endomorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$.*
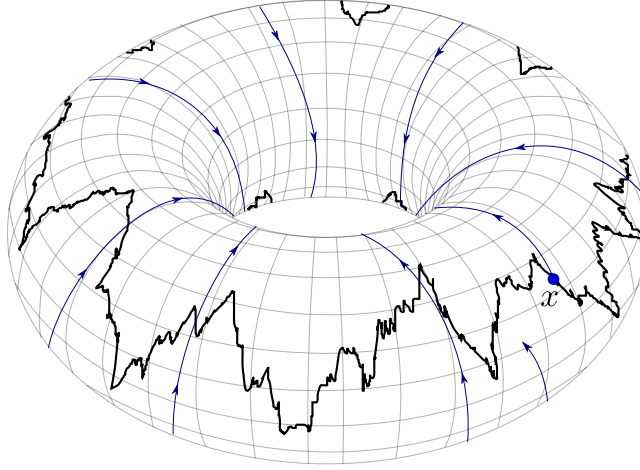
FIGURE 2. A realisation of the graph of Brownian motion $W_t$ embedded onto the 2-torus $\mathbb{T}^2$ with the beginning of a $T$-orbit of $x = (t, W_t)$.

1.5. **Organisation of the paper.** In Section 2 we give the necessary background from Itō calculus. In Section 3 we will give the proof of our main result Theorem 1.3. The key estimates are obtained in Section 3.2 and Section 3.3 corresponding to the two cases discussed above. In Section 4 we prove the equidistribution Theorem 1.6 and finally in Section 5 we discuss further prospects for this work.

## 2. ITŌ CALCULUS

2.1. **Stochastic integration.** In the proof of the main Theorem 1.3 we will end up studying integrals of the form $\int f(X_t)\, dt$ for some stochastic processes $X_t$ and smooth scalar functions $f$. As standard analysis methods cannot be applied to these integrals, we need theory from stochastic analysis. Stochastic analysis provides a pleasant framework to deal with non-smooth processes, like the Wiener process $W_t$, but still preserves much of the analytic tools available in smooth analysis. In this section we discuss the specific tools from *Itō calculus* which we will rely on, the main references for this section are given in the book [21].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ be a filtered probability space, that is, $\mathcal{F}_t \subset \mathcal{F}$ is an increasing filtration in $t$. Let $W = W_t$ be the Wiener process adapted to this filtered probability space, that is, $W_t$ is $\mathcal{F}_t$ measurable and for each $t, s \geqslant 0$ the increment $W_{t+s} - W_t$ is independent of $\mathcal{F}_t$. We say that an $\mathbb{R}$ or $\mathbb{C}$ valued stochastic process $Z_t$ is *adapted* if it is $\mathcal{F}_t$ measurable for all $t \geqslant 0$. We will say that a real or complex valued adapted process $Z_t$ is $W_t$ *integrable* if the *quadratic variation* $\int_0^T |Z_t|^2\, dt < \infty$ for any time $T \geqslant 0$. Given a real valued adapted $W_t$ integrable stochastic process $X_t$, then $\mathbb{P}$ almost surely for any time $T \geqslant 0$ it is possible to construct a *stochastic integral*

$$\int_0^T X_t\, dW_t$$

of $X_t$ with respect to $W_t$ in the sense of Itō, see [21, Chapter 3.2]. We use the differential notation $dU_t = X_t\, dW_t$ to mean that $\mathbb{P}$ almost surely $U_T - U_0$ is the stochastic integral of $X_t$ with respect to $W_t$ at time $T \geqslant 0$.

We mainly deal with complex valued stochastic processes so for the sake of convenience we will also define the *complex valued* stochastic integral for a $\mathbb{C}$ valued $W_t$ integrable adapted process $Z_t$ is defined coordinate-wise using real integrals:

$$\int_0^T Z_t \, dW_t := \int_0^T \operatorname{Re} Z_t \, dW_t + i \int_0^T \operatorname{Im} Z_t \, dW_t,$$

where the real integrals are standard $\mathbb{R}$ valued stochastic integrals with respect to the Wiener process $W_t$. We write $dZ_t := dX_t + idY_t$ for a complex valued process $Z_t = X_t + iY_t$ with $\mathbb{R}$ valued $X_t$ and $Y_t$.

2.2. **Itō drift-diffusion processes.** The main class of adapted processes where Itō calculus is applied to and which will appear in our proof of Theorem 1.3 are given by Wiener processes with drift and diffusion coefficients. These are called Itō drift-diffusions:

**Definition 2.1** (Itō drift-diffusion process)**.** A real or complex valued adapted stochastic process $X_t$ is called a *Itō drift-diffusion process* if there exists a Lebesgue integrable adapted $b_t$ and $W_t$ integrable adapted $\sigma_t$ such that $X_t$ satisfies the stochastic differential equation

$$dX_t = b_t \, dt + \sigma_t \, dW_t.$$

For Itō drift-diffusion processes there exists the following important analogue of the change of variable formula, which follows from robustness of Taylor expansions for stochastic differentials:

**Lemma 2.2** (Itō's lemma)**.** *Let $X_t$ be an Itō drift-diffusion process and $f : \mathbb{R} \to \mathbb{R}$ twice differentiable. Then $f(X_t)$ is an Itō drift-diffusion process such that $\mathbb{P}$ almost surely for any $T \geqslant 0$ we have*

$$f(X_T) - f(X_0) = \int_0^T \left( b_t f'(X_t) + \frac{\sigma_t^2}{2} f''(X_t) \right) dt + \int_0^T \sigma_t f'(X_t) \, dW_t.$$

Itō's lemma in this form was given in this pathwise form in [21, Theorem 3.3]. By using the definition of complex valued stochastic integral, we can also write a complex valued Itō's lemma as follows:

**Lemma 2.3** (Complex Itō's lemma)**.** *Let $X_t$ be a Itō drift-diffusion process and $f : \mathbb{R} \to \mathbb{C}$ twice differentiable. Then $f(X_t)$ is an Itō drift-diffusion process such that for $\mathbb{P}$ almost surely for any $T \geqslant 0$ we have*

$$f(X_T) - f(X_0) = \int_0^T \left( b_t f'(X_t) + \frac{\sigma_t^2}{2} f''(X_t) \right) dt + \int_0^T \sigma_t f'(X_t) \, dW_t.$$

*Proof.* We can write $f = f_1 + if_2$ for real valued twice differentiable $f_1, f_2 : \mathbb{R} \to \mathbb{R}$. Then the derivatives $f' = f_1' + if_2'$ and $f'' = f_1'' + if_2''$. Moreover, by Itō's lemma (Lemma 2.2) we obtain for each $j = 1, 2$ that

$$df_j(X_t) = \left( b_t f_j'(X_t) + \frac{\sigma_t^2}{2} f_j'(X_t) \right) dt + \sigma_t f_j'(X_t) \, dW_t.$$

Then by the convention $df(X_t) = df_1(X_t) + idf_2(X_t)$ this gives

$$df(X_t) = \left( b_t f'(X_t) + \frac{\sigma_t^2}{2} f''(X_t) \right) dt + \sigma_t f'(X_t) \, dW_t$$

as required. □

2.3. **Moment estimation.** Itō's lemma allows us to pass from integrals of the form $\int_0^T f(X_t)\,dt$ to $\int_0^T g(X_t)\,dW_t$ for functions $g$ obtained from derivatives of $f$. In our case we will end up trying to understand the higher moments of the stochastic integrals $\int_0^T g(X_t)\,dW_t$, which will tell us about the distribution of these integrals. A very standard tool to compute the moments in Itō calculus are the *Itō isometry* and more general *Burkholder-Davis-Gundy inequalities* (see [5]), which allows to pass from stochastic integrals to their quadratic variations (that just involve Lebesgue integral). Here we state the following version:

**Lemma 2.4** (Burkholder-Davis-Gundy inequality). *Let $X_t$ be a real valued $W_t$ integrable adapted process. Then for all $1 \leqslant p < \infty$ we have*

$$\mathbb{E}\Big[\Big(\sup_{0\leqslant s\leqslant 1}\Big|\int_0^s X_t\,dW_t\Big|\Big)^{2p}\Big] \leqslant 2\sqrt{10p}\,\mathbb{E}\Big[\Big(\int_0^1 X_t^2\,dt\Big)^p\Big].$$

This was given by Peskir [32] with the constant $2\sqrt{10p}$.

## 3. Proof of the main result

3.1. **Preliminaries and overview of the proof.** Let us now review how we will prove (1.2) and thus Theorem 1.3. Fix $\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2$ with modulus $u > 0$ and argument $\theta \in [0, 2\pi)$. Notice that by the definition of the graph measure $\mu$, the Fourier transform has the form

$$\widehat{\mu}(\xi) = \int_0^1 \exp(iX_t)\,dt,$$

where $X_t$ is the real valued stochastic process

$$X_t := -2\pi u(t\cos\theta + W_t\sin\theta). \tag{3.1}$$

The first observation is that $X_t$ is an adapted $W_t$ integrable process and in fact an Itō drift-diffusion process (recall Definition 2.1) satisfying

$$dX_t = b\,dt + \sigma\,dW_t$$

for deterministic and time independent coefficients $b = -2\pi u\cos\theta$ and $\sigma = -2\pi u\sin\theta$. The proof of bounding $\widehat{\mu}(\xi)$ will heavily depend on the value of the angle $\theta$ we have for $\xi$ and in particular how close the determining angle $\theta$ is to $0$, $\pi$ or $2\pi$ is with respect to $u^{-1/2}$. For this purpose, we define the notions of *horizontal* and *vertical* angles:

**Definition 3.1** (Horizontal and vertical angles). Define the threshold angle

$$\theta_u := \min\{u^{-1/2}, \tfrac{\pi}{4}\}.$$

Partition the angles $[0, 2\pi)$ using $\theta_u$ into the *horizontal angles*

$$H_u := [0, \theta_u] \cup [\pi - \theta_u, \pi + \theta_u] \cup [2\pi - \theta_u, 2\pi).$$

and the *vertical angles*

$$V_u := [0, 2\pi) \setminus H_u.$$

In other words $H_u$ contains the $\theta_u$ neighborhoods of $0$ and $\pi$ on the circle mod $2\pi$ and $V_u$ the $\pi/2 - \theta_u$ neighborhoods of $\pi/2$ and $3\pi/2$ respectively, see Figure 3.1.
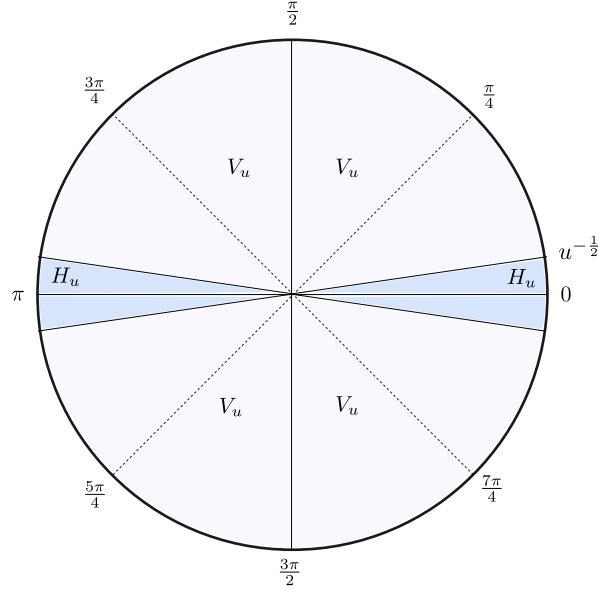
FIGURE 3. Splitting of $[0, 2\pi)$ to horizontal angles $H_u$ and the vertical angles $V_u$

The proof will split into two cases in Sections 3.2 and 3.3 for bounding the Fourier transform $\widehat{\mu}(\xi)$ depending on whether $\theta \in H_u$ or $\theta \in V_u$.

(1) Section 3.2 concerns angles $\theta \in H_u$, that is, close to horizontal directions $0$ or $\pi$ and as mentioned in the introduction our main hope here is that the smallness (with respect to $u^{-1/2}$) of the diffusion component $bW_t$ will help us in transferring the decay of Lebesgue measure to the decay of $\widehat{\mu}$. This is where in particular Itō's lemma (see Lemma 2.3) becomes crucial as it can be applied to the process $f(X_t)$ with the function $f(x) = \exp(ix)$.

(2) Section 3.3 handles the angles $\theta \in V_u$ and here the plan is to use the fact that we are $u^{-1/2}$ bounded away from horizontal angles to ignore the drift component $bt$ of the drift-diffusion process $X_t$ and apply Kahane's bound for these directions. This turns out to be possible due to a representation of the higher moments Kahane obtained in his result on Brownian images.

It turns out that in both of the Sections 3.2 and 3.3 we only obtain decay of the Fourier transform $\widehat{\mu}(\mathbf{k})$ for $\mathbf{k}$ in an $\varepsilon$-grid $\varepsilon\mathbb{Z}^2$ for all small $\varepsilon > 0$. Here the randomness will depend on $\varepsilon > 0$ but thanks to an argument also used by Kahane in [19], one can pass from this information to the full decay almost surely. See Section 3.4 for the details.

Let us now proceed to bound $|\widehat{\mu}(\xi)|$. In both Sections 3.2 and 3.3 below we will end up bounding trigonometric functions with respect to $\theta_u$ and for this purpose we will need the following standard bounds:

**Lemma 3.2** (Trigonometric bounds). *We have the following bounds:*

(1) *If $\theta \in H_u$, then*
$$|\sin\theta| \leqslant u^{-1/2} \quad and \quad |\cos\theta| \geqslant \tfrac{1}{\sqrt{2}}.$$

(2) If $\theta \in V_u$, then
$$|\sin\theta| \geqslant \min\{\tfrac{2}{\pi}u^{-1/2}, \tfrac{1}{\sqrt{2}}\}.$$

*Proof.* For $\alpha \in [0, \frac{\pi}{2}]$ we have that both cos and sin are non-negative. Moreover, here $\frac{2}{\pi}\alpha \leqslant \sin\alpha \leqslant \alpha$. Thus for $\theta \in [0, \theta_u]$ we have
$$\sin\theta \leqslant \theta \leqslant \theta_u \leqslant u^{-1/2} \quad \text{and} \quad \cos\theta \geqslant \cos\theta_u \geqslant \cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}.$$
and for $\theta \in (\theta_u, \frac{\pi}{2}]$ as $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ we obtain
$$\sin\theta \geqslant \min\{\tfrac{2}{\pi}u^{-1/2}, \tfrac{1}{\sqrt{2}}\}.$$

This gives the claim as we may reduce the estimates back to the estimates for $\theta \in [0, \frac{\pi}{2}]$ by using standard invariance identities for sin and cos. $\qquad\square$

3.2. **Horizontal angles.** When $\theta \in H_u$ we will first obtain the following estimate on $\varepsilon$-grids:

**Lemma 3.3.** *Fix $\varepsilon > 0$. Almost surely there exists a random constant $C_\omega > 0$ such that for any $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ with $\theta \in H_u$ we have*
$$|\widehat{\mu}(\mathbf{k})| \leqslant C_\omega |\mathbf{k}|^{-1/2}.$$

Given $\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\}$ and a realisation $(W_t)$ define a random time $T = T_\omega(\xi) \in [0, 1]$, which the minimum value of $t \in [0, 1]$ such that
$$X_t = \begin{cases} -2\pi\lceil u(\cos\theta + W_1\sin\theta)\rceil, & \text{if } X_1 \geqslant 0; \\ -2\pi\lfloor u(\cos\theta + W_1\sin\theta)\rfloor, & \text{if } X_1 < 0. \end{cases}$$

Such a time $T$ exists almost surely since $X_0 = 0$ and $X_t$ is almost surely continuous (since $W_t$ is almost surely continuous). Splitting the integral of $Z_t$ up into 'complete rotations' and 'what is left over', one obtains
$$\int_0^1 Z_t\, dt = \int_0^T Z_t\, dt + \int_T^1 Z_t\, dt.$$
For the integral over $[T, 1]$ we get the following estimate.

**Lemma 3.4.** *Almost surely there exists a random constant $C_\omega > 0$ such that for any $\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\}$ with $\theta \in H_u$ we have*
$$\left| \int_T^1 Z_t\, dt \right| \leqslant C_\omega |\xi|^{-1/2}.$$

*Proof.* Since $W_t$ is almost surely continuous, there almost surely exists a random constant $M_\omega > 1$ such that $W_t \in [-M_\omega, M_\omega]$ for all $t \in [0, 1]$. Define the real-valued process
$$Y_t := u(t\cos\theta + W_t\sin\theta)$$
so $X_t = -2\pi Y_t$. Suppose $X_1 \geqslant 0$. In this case $Y_T = \lceil Y_1 \rceil \leqslant 0$ and so $Y_1 + 1 \geqslant Y_T \geqslant Y_1$. Moreover, when $X_1 < 0$ we have $Y_T = \lfloor Y_1 \rfloor > 0$ and $Y_1 \geqslant Y_T \geqslant Y_1 - 1$. Thus no matter what is the sign of $X_1$ is, we always have almost surely
$$u(\cos\theta + W_1\sin\theta) + 1 \geqslant u(T\cos\theta + W_T\sin\theta) \geqslant u(\cos\theta + W_1\sin\theta) - 1.$$

Therefore, in the case $\cos\theta > 0$ we obtain

$$T \geqslant 1 + W_1 \frac{\sin\theta}{\cos\theta} - W_T \frac{\sin\theta}{\cos\theta} - \frac{1}{u\cos\theta}$$

and when $\cos\theta < 0$ we have

$$T \geqslant 1 + W_1 \frac{\sin\theta}{\cos\theta} - W_T \frac{\sin\theta}{\cos\theta} + \frac{1}{u\cos\theta}.$$

Since $u \in H_u$ Lemma 3.2 together with $W_t \in [-M_\omega, M_\omega]$ yields

$$T \geqslant 1 - 2\sqrt{2}M_\omega u^{-1/2} - \frac{\sqrt{2}}{u}.$$

Recalling $M_\omega > 1$ this gives

$$\left| \int_T^1 Z_t\,dt \right| \leqslant \int_T^1 |Z_t|\,dt = 1 - T \leqslant 4M_\omega u^{-1/2}$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now estimate the integral over $[0, T]$, which is where Itō calculus comes into play.

**Lemma 3.5.** *Fix $\varepsilon > 0$. Almost surely there exists a random constant $C_\omega > 0$ such that for any $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ with $\theta \in H_u$ we have*

$$\left| \int_0^T Z_t\,dt \right| \leqslant C_\omega |\mathbf{k}|^{-1/2}.$$

To prove Lemma 3.5, we first need to compute the higher order moments of the random variable $\int_0^T Z_t\,dt$.

**Lemma 3.6.** *For any $p \in \mathbb{N}$ and $\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\}$ with $\theta \in H_u$ the $2p$th moment*

$$\mathbb{E}\left| \int_0^T Z_t\,dt \right|^{2p} \leqslant 13p^{1/2}\,4^p|\xi|^{-p}.$$

*Proof.* Recall that

$$X_t = -2\pi u(t\cos\theta + W_t\sin\theta)$$

is an Itō drift-diffusion process satisfying the stochastic differential equation

$$dX_t = b\,dt + \sigma\,dW_t$$

for deterministic and time independent coefficients $b = -2\pi u\cos\theta$ and $\sigma = -2\pi u\sin\theta$. Writing

$$f(x) := \exp(ix), \quad x \in \mathbb{R},$$

then $Z_t = f(X_t)$, $f'(x) = i\exp(ix)$ and $f''(x) = -\exp(ix)$. Thus by complex Itō's lemma (see Lemma 2.3) we have $\mathbb{P}$ almost surely

$$f(X_T) - f(X_0) = (bi - \sigma^2/2)\int_0^T f(X_t)\,dt + \sigma i \int_0^T f(X_t)\,dW_t. \qquad (3.2)$$

Note that $T_\omega \leqslant 1$ is random and only $\mathcal{F}_1$ measurable, thus it is not a stopping time. However, as Lemma 2.3 is given *pathwise*, that is, $\mathbb{P}$ almost surely Itō's lemma holds for any time $T \geqslant 0$

then as $T_\omega$ is $\mathbb{P}$ almost surely well-defined, we have (3.2) almost surely. Since $X_0$ and $X_T$ are $2\pi$ multiples of integers by definition, we have $f(X_T) = f(X_0) = 1$. Thus (3.2) gives

$$\int_0^T f(X_t)\,dt = -\frac{\sigma i}{bi - \sigma^2/2} \int_0^T f(X_t)\,dW_t$$

Since $b$ and $\sigma$ are deterministic, this yields that the $2p$th moment

$$\mathbb{E}\left| \int_0^T f(X_t)\,dt \right|^{2p} = \left| \frac{\sigma i}{bi - \sigma^2/2} \right|^{2p} \mathbb{E}\left| \int_0^T f(X_t)\,dW_t \right|^{2p}.$$

Applying the Burkholder–Davis–Gundy inequality (see Lemma 2.4) for the process $\cos X_t$ gives

$$\mathbb{E}\left[ \left| \int_0^T \cos X_t\,dW_t \right|^{2p} \right] \leqslant \mathbb{E}\left[ \left( \sup_{0 \leqslant s \leqslant 1} \left| \int_0^s \cos X_t\,dW_t \right| \right)^{2p} \right]$$

$$\leqslant 2\sqrt{10p}\,\mathbb{E}\left( \int_0^1 \cos^2 X_t\,dt \right)^p$$

$$\leqslant 2\sqrt{10p}$$

since $\cos^2 \leqslant 1$. Similar application for the process $\sin X_t$ gives

$$\mathbb{E}\left[ \left| \int_0^T \sin X_t\,dW_t \right|^{2p} \right] \leqslant 2\sqrt{10p}.$$

By Euler's formula, we can write $f(X_t) = \cos X_t + i \sin X_t$ and so

$$\int_0^T f(X_t)\,dW_t = \int_0^T \cos X_t\,dW_t + i \int_0^T \sin X_t\,dW_t.$$

Hence

$$\mathbb{E}\left| \int_0^T f(X_t)\,dW_t \right|^{2p} = \mathbb{E}\left[ \left( \left| \int_0^T \cos X_t\,dW_t \right|^2 + \left| \int_0^T \sin X_t\,dW_t \right|^2 \right)^p \right]$$

$$\leqslant \mathbb{E}\left[ 2^p \left| \int_0^T \cos X_t\,dW_t \right|^{2p} + 2^p \left| \int_0^T \sin X_t\,dW_t \right|^{2p} \right]$$

$$= 2^p \left( \mathbb{E}\left[ \left| \int_0^T \cos X_t\,dW_t \right|^{2p} \right] + \mathbb{E}\left[ \left| \int_0^T \sin X_t\,dW_t \right|^{2p} \right] \right)$$

$$\leqslant 2^p 4\sqrt{10p}.$$

Moreover, as $\theta \in H_u$ we have by Lemma 3.2 that $\cos^2 \theta \geqslant 1/2$ and $\sin^2 \theta \leqslant u^{-1}$. Hence

$$\left| \frac{\sigma i}{bi - \sigma^2/2} \right|^2 = \frac{\sigma^2}{b^2 + \sigma^4/4} \leqslant \frac{\sigma^2}{b^2} = \frac{4\pi^2 u^2 \sin^2 \theta}{4\pi^2 u^2 \cos^2 \theta} = \frac{\sin^2 \theta}{\cos^2 \theta} \leqslant 2u^{-1}.$$

Therefore,

$$\mathbb{E}\left| \int_0^T f(X_t)\,dt \right|^{2p} \leqslant 4\sqrt{10p}\,4^p \theta^{2p} \leqslant 13 p^{1/2}\,4^p u^{-p}.$$

as required. $\qquad\square$

*Proof of Lemma 3.5.* Fix $\varepsilon > 0$. Then for all $\mathbf{k} \in \varepsilon \mathbb{Z}^2 \setminus \{0\}$ define the random variable

$$I(\mathbf{k}) := \left( \int_0^T Z_t \, dt \right) \cdot \chi_A(\mathbf{k}),$$

where $\chi_A$ is the indicator function on the set

$$A := \{\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\} : \theta \in H_u\}.$$

Note that $I(\mathbf{k})$ is well-defined and finite since $|\int_0^T Z_t \, dt| \leqslant 1$ by $|\exp(ix)| = 1$ property. Lemma 3.6 now yields for any $\mathbf{k} \in \varepsilon \mathbb{Z}^2 \setminus \{0\}$ and $p \in \mathbb{N}$ that

$$\mathbb{E}|I(\mathbf{k})|^{2p} \leqslant 13 p^{1/2} \, 4^p |\mathbf{k}|^{-p}$$

as when $\mathbf{k} \notin A$ we have $I(\mathbf{k}) \equiv 0$. Write $p_{\mathbf{k}} = \lfloor \log|\mathbf{k}| \rfloor$. Then

$$\mathbb{E} \sum_{\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}} |\mathbf{k}|^{-3} \frac{|I(\mathbf{k})|^{2p_{\mathbf{k}}}}{13 p_{\mathbf{k}}^{1/2} \, 4^{p_{\mathbf{k}}} |\mathbf{k}|^{-p_{\mathbf{k}}}} \leqslant \sum_{\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}} |\mathbf{k}|^{-3} < \infty.$$

This means that the summands tend to 0 almost surely as $|\mathbf{k}| \to \infty$ and so we can find a random constant $C_\omega > 0$ such that for all $\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ we have

$$|\mathbf{k}|^{-3} \frac{|I(\mathbf{k})|^{2p_{\mathbf{k}}}}{13 p_{\mathbf{k}}^{1/2} \, 4^{p_{\mathbf{k}}} |\mathbf{k}|^{-p_{\mathbf{k}}}} \leqslant C_\omega.$$

Therefore, by possibly making $C_\omega$ bigger we obtain

$$|I(\mathbf{k})| \leqslant C_\omega |\mathbf{k}|^{-1/2}.$$

This holds for each $\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ so by definition of $I(\mathbf{k})$ we have whenever $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ with $\theta \in H_u$ that

$$\left| \int_0^T Z_t \, dt \right| \leqslant C_\omega u^{-1/2}$$

as claimed. $\qquad\square$

We are now in position to complete the proof of Lemma 3.3.

*Proof of Lemma 3.3.* Fix $\varepsilon > 0$. By the splitting

$$\widehat{\mu}(\xi) = \int_0^1 Z_t \, dt = \int_0^T Z_t \, dt + \int_T^1 Z_t \, dt$$

and Lemmas 3.4 and 3.5, we have that almost surely there exists a constant $C_\omega > 0$ such that for all $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ with $\theta \in H_u$ we have

$$|\widehat{\mu}(\mathbf{k})| \leqslant \left| \int_0^T Z_t \, dt \right| + \left| \int_T^1 Z_t \, dt \right| \leqslant C_\omega |\mathbf{k}|^{-1/2}$$

as required. $\qquad\square$

3.3. **Vertical angles.** In this section we apply Kahane's work to obtain Fourier decay estimates when $\theta \in V_u$.

**Lemma 3.7.** *Fix $\varepsilon > 0$. Almost surely there exists a random constant $C_\omega > 0$ such that for any $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ with $\theta \in V_u$ we have*

$$|\widehat{\mu}(\mathbf{k})| \leqslant C_\omega |\mathbf{k}|^{-1/2} \sqrt{\log |\mathbf{k}|}.$$

Let us discuss a few estimates Kahane obtained in [19]. Let $\nu$ be the push-forward of Lebesgue measure on $[0, 1]$ under the map $t \mapsto W_t$, that is, $\nu$ is the Brownian image of Lebesgue measure. Kahane established the following:

**Theorem 3.8** (Kahane, page 255, [19]). *Almost surely*

$$|\widehat{\nu}(v)| \leqslant O(|v|^{-1}\sqrt{\log |v|}) \quad \text{as } |v| \to \infty.$$

The key ingredient for the proof of Theorem 3.8 was based on establishing the following bound for the higher moments:

**Lemma 3.9** (Kahane, page 254, [19], estimate (2)). *There exists a constant $C > 0$ such that for any $v \in \mathbb{R} \setminus \{0\}$ and any $p \in \mathbb{N}$ we have*

$$\mathbb{E}|\widehat{\nu}(v)|^{2p} \leqslant C^p p^p |v|^{-2p}.$$

We can use Lemma 3.9 to give a bound on the higher moments in our setting, but with the price that the exponent will increase from $-2p$ to $-p$.

**Lemma 3.10.** *There exists a constant $C > 0$ such that for any $p \in \mathbb{N}$ and $\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\}$ with $\theta \in V_u$ the $2p$th moment satisfies*

$$\mathbb{E}|\widehat{\mu}(\xi)|^{2p} \leqslant C^p p^p |\xi|^{-p}.$$

*Proof.* Write $\mathbf{t} = (t_1, \ldots, t_p) \in [0, 1]^p$ and $d\mathbf{t}$ as the Lebesgue measure on $[0, 1]^p$. Given $\mathbf{t}, \mathbf{s} \in [0, 1]^p$, we denote

$$\varphi(\mathbf{t}, \mathbf{s}) := \sum_{k=1}^p (t_k - s_k), \quad \psi(\mathbf{t}, \mathbf{s}) := \sum_{k=1}^p (W_{t_k} - W_{s_k}), \quad \text{and} \quad \Psi(\mathbf{t}, \mathbf{s}) := \mathbb{E}|\varphi(\mathbf{t}, \mathbf{s})|^2.$$

By the definition of $\mu_\theta$, $\mu$ and the Fourier-transform, and using the fact that the multivariate process

$$X(\mathbf{t}, \mathbf{s}) := -2\pi \cos(\theta)\varphi(\mathbf{t}, \mathbf{s}) - 2\pi \sin(\theta)\psi(\mathbf{t}, \mathbf{s})$$

is Gaussian with mean $-2\pi\cos(\theta)\varphi(\mathbf{t}, \mathbf{s})$ and variance $4\pi^2 \sin^2(\theta)\Psi(\mathbf{t}, \mathbf{s})$, we have through Fubini's theorem and the formula for the characteristic function that

$$
\begin{aligned}
\mathbb{E}|\widehat{\mu}(\xi)|^{2p} &= \mathbb{E}\int_{[0,1]^p}\int_{[0,1]^p} \exp(-2\pi i u(\cos(\theta)\varphi(\mathbf{t}, \mathbf{s}) + \sin(\theta)\psi(\mathbf{t}, \mathbf{s})))\, d\mathbf{t}\, d\mathbf{s} \\
&= \int_{[0,1]^p}\int_{[0,1]^p} \mathbb{E}\exp(iuX(\mathbf{t}, \mathbf{s}))\, d\mathbf{t}\, d\mathbf{s} \\
&= \int_{[0,1]^p}\int_{[0,1]^p} \exp(-2\pi i \cos(\theta)u\varphi(\mathbf{t}, \mathbf{s}) - 2\pi^2|u\sin(\theta)|^2\Psi(\mathbf{t}, \mathbf{s}))\, d\mathbf{t}\, d\mathbf{s}.
\end{aligned}
$$

Thus by taking absolute values inside the integrals, and observing that $|\exp(ix)| = 1$ for any $x \in \mathbb{R}$, we obtain

$$\mathbb{E}|\widehat{\mu}(\xi)|^{2p} \leqslant \int_{[0,1]^p} \int_{[0,1]^p} \exp(-2\pi^2|u\sin(\theta)|^2 \Psi(\mathbf{t},\mathbf{s})) \, d\mathbf{t} \, d\mathbf{s}. \tag{3.3}$$

On the other hand, by doing the expansion again for the Fourier transform $\widehat{\nu}$ of the image measure $\nu$ at $v := u\sin(\theta) \in \mathbb{R} \setminus \{0\}$ we see that

$$\mathbb{E}|\widehat{\nu}(v)|^{2p} = \mathbb{E} \int_{[0,1]^p} \int_{[0,1]^p} \exp(-2\pi i v \psi(\mathbf{t},\mathbf{s})) \, d\mathbf{t} \, d\mathbf{s} = \int_{[0,1]^p} \int_{[0,1]^p} \exp(-2\pi^2 v^2 \Psi(\mathbf{t},\mathbf{s})) \, d\mathbf{t} \, d\mathbf{s},$$

which equals to (3.3). Thus by Lemma 3.9 we have

$$\mathbb{E}|\widehat{\mu}(\xi)|^{2p} \leqslant C^p p^p |v|^{-2p}.$$

Since $\theta \in V_u$ we have $|\sin\theta| \geqslant \min\{\frac{2}{\pi}u^{-1/2}, \frac{1}{\sqrt{2}}\}$. When $|\sin\theta| \geqslant \frac{1}{\sqrt{2}}$ we obtain

$$C^p p^p |v|^{-2p} \leqslant (2C)^p p^p u^{-2p} \leqslant (2C)^p p^p u^{-p}.$$

On the other hand, if $|\sin\theta| \geqslant \frac{2}{\pi}u^{-1/2}$ we have

$$C^p p^p |v|^{-2p} \leqslant C^p p^p (2u^{-1/2}/\pi)^{-2p} u^{-2p} \leqslant (C\pi^2/4)^p p^p u^{-p}$$

This completes the proof. $\qquad\square$

Now we can complete the proof of Lemma 3.7 for vertical directions:

*Proof of Lemma 3.7.* Fix $\varepsilon > 0$. Then for all $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ define the random variable

$$F(\mathbf{k}) := \widehat{\mu}(\mathbf{k})\chi_B(\mathbf{k}),$$

where

$$B := \{\xi = u(\cos\theta, \sin\theta) \in \mathbb{R}^2 \setminus \{0\} : \theta \in V_u\}.$$

Now $F(\mathbf{k})$ is a well-defined finite random variable as $|\widehat{\mu}(\mathbf{k})| \leqslant 1$ for any $\mathbf{k}$. From Lemma 3.10 we obtain for any $\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ and $p \in \mathbb{N}$ that

$$\mathbb{E}|F(\mathbf{k})|^{2p} \leqslant C^p p^p |\mathbf{k}|^{-p}.$$

Write $p_{\mathbf{k}} = \lfloor \log|\mathbf{k}| \rfloor$. Then

$$\mathbb{E} \sum_{\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}} |\mathbf{k}|^{-3} \frac{|F(\mathbf{k})|^{2p_{\mathbf{k}}}}{C^{p_{\mathbf{k}}} p_{\mathbf{k}}^{p_{\mathbf{k}}} |\mathbf{k}|^{-p_{\mathbf{k}}}} \leqslant \sum_{\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}} |\mathbf{k}|^{-3} < \infty.$$

This means that the summands tend to 0 almost surely as $|\mathbf{k}| \to \infty$ and so we can find a random constant $C_\omega > 0$ such that for all $\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ we have

$$|\mathbf{k}|^{-3} \frac{|F(\mathbf{k})|^{2p_{\mathbf{k}}}}{C^{p_{\mathbf{k}}} p_{\mathbf{k}}^{p_{\mathbf{k}}} |\mathbf{k}|^{-p_{\mathbf{k}}}} \leqslant C_\omega.$$

Thus possibly making $C_\omega$ bigger this yields

$$|F(\mathbf{k})| \leqslant C_\omega |\mathbf{k}|^{-1/2} \sqrt{\log|\mathbf{k}|}.$$

Now this holds for each $\mathbf{k} \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ so by definition of $F(\mathbf{k})$ we have, whenever $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ with $\theta \in V_u$, that

$$|\widehat{\mu}(\mathbf{k})| \leqslant C_\omega |\mathbf{k}|^{-1/2} \sqrt{\log|\mathbf{k}|}$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.4. **From lattices to $\mathbb{R}^2$.** We can now complete the proof of the main theorem. For this purpose, we need the following comparison lemma used by Kahane that allows to pass from convergence on lattices for Fourier transform to the whole space:

**Lemma 3.11** (Kahane, Lemma 1, page 252, [19]). *Suppose $\tau$ is a measure on $\mathbb{R}^2$ with support in $(-1,1)^2$. Suppose $\varphi, \psi : (0,\infty) \to (0,\infty)$ that are decreasing as $t \to \infty$ with the doubling properties*

$$\varphi(t/2) = O(\varphi(t)) \quad and \quad \psi(t/2) = O(\psi(t)) \quad as\ t \to \infty.$$

*If the Fourier transform of $\tau$ along the integer lattice $\mathbb{Z}^2$ satisfies*

$$|\widehat{\tau}(\mathbf{n})| = O(\varphi(|\mathbf{n}|)/\psi(|\mathbf{n}|)), \quad as\ |\mathbf{n}| \to \infty,$$

*then*

$$|\widehat{\tau}(\xi)| = O(\varphi(|\xi|)/\psi(|\xi|)), \quad as\ |\xi| \to \infty.$$

*Proof of Theorem 1.3.* Combining Lemmas 3.7 and 3.3 we have that for any $\varepsilon > 0$, almost surely, there exists some random constant $C_\omega > 0$ such that for any $\mathbf{k} = u(\cos\theta, \sin\theta) \in \varepsilon\mathbb{Z}^2 \setminus \{0\}$ we have

$$|\widehat{\mu}(\mathbf{k})| \leqslant C_\omega |\mathbf{k}|^{-1/2} \sqrt{\log|\mathbf{k}|}. \qquad\qquad\qquad\qquad (3.4)$$

Define a measure $\tau_\varepsilon$ on $\mathbb{R}^2$ such that

$$\widehat{\tau}_\varepsilon(\xi) := \widehat{\mu}(\varepsilon\xi), \quad \xi \in \mathbb{R}^2.$$

By the almost sure continuity of $W_t$, we have that there exists a random constant $M_\omega > 0$ such that the diameter of the support of $\mu$ is at most $M_\omega$ almost surely. Taking an intersection of the events that (3.4) holds for $\varepsilon = 1/n$ over all $n \in \mathbb{N}$ allows us to find a random $\varepsilon = \varepsilon_\omega > 0$ such that $\mu$ is supported on a set of diameter strictly less than $1/\varepsilon$ and (3.4) holds almost surely with this $\varepsilon$. This guarantees that the measure $\tau_\varepsilon$ is supported on $(-1,1)^2$ and so applying Lemma 3.11 with the measure $\tau = \tau_\varepsilon$ and the maps $\varphi(t) := \sqrt{\log t}$ and $\psi(t) := t^{1/2}$ gives the claim. $\quad\square$

## 4. PROOF OF THE EQUIDISTRIBUTION THEOREM

The strategy of the proof of Theorem 1.6 is similar to that implemented by Queffelec and Ramare [33] for the one dimensional version of Davenport-Erdös-LeVeque. For this purpose we need the following Lemma.

**Lemma 4.1** (Lemme 7.3 [33]). *Suppose $(r_N) \subset (0,\infty)$ be a sequence of reals such that $\sum_{N=1}^\infty \frac{r_N}{N} < \infty$. Then there is a subsequence $N_j \to \infty$, $j \in \mathbb{N}$, such that*

$$\sum_{j=1}^\infty r_{N_j} < \infty \quad and \quad \lim_{j \to \infty} \frac{N_{j+1}}{N_j} = 1.$$

*Proof of Theorem 1.6.* Let $\mu$ be a measure on the $d$-torus $\mathbb{T}^d$ for which there exists some $\alpha > 0$ and $C > 0$ such that the Fourier transform satisfies

$$|\widehat{\mu}(\xi)| \leqslant C|\xi|^{-\alpha}$$

for any $\xi \in \mathbb{R}^2 \setminus \{0\}$. Also, let $T : \mathbb{T}^d \to \mathbb{T}^d$ an expanding toral endomorphism. Fix $\mathbf{k} \in \mathbb{Z}^d \setminus \{0\}$ and for $N \in \mathbb{N}$ write

$$S_N(x) := \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i \mathbf{k} \cdot T^n(x)),$$

where $\mathbf{k}$ has been suppressed from the notation. By Weyl's equidistribution criterion (Theorem 1.5) for our claim it is enough to prove that $S_N(x) \to 0$ as $N \to \infty$ for $\mu$ almost every $x$. Note that from this one can deduce that the result holds simultaneously for *all* expanding toral endomorphisms simultaneously follows since such maps are represented by two dimensional matrices with integer coefficients and so there are only countably many possibilities.

Write

$$r_N := \int |S_N(x)|^2 \, d\mu(x),$$

and

$$T(x) = Ax \mod 1, \quad x \in \mathbb{T}^d,$$

for some $A \in \mathbb{Z}^{d \times d}$. Also write $A^* \in \mathbb{Z}^{d \times d}$ for the adjoint of $A$. We have for each $x \in \mathbb{T}^d$ that

$$\mathbf{k} \cdot T(x) = \mathbf{k} \cdot A(x) - \mathbf{k} \cdot \lfloor A(x) \rfloor = A^*(\mathbf{k}) \cdot x - \mathbf{k} \cdot \lfloor A(x) \rfloor, \tag{4.1}$$

where $\lfloor y \rfloor \in \mathbb{Z}^d$ of $y \in \mathbb{R}^d$ is the vector with entries $\lfloor y_j \rfloor$ for each coordinate $j = 1, \ldots, d$. Hence as $\mathbf{k} \in \mathbb{Z}^d$ the number $\mathbf{k} \cdot \lfloor A(x) \rfloor \in \mathbb{Z}$. Now as $\mathbf{k} \in \mathbb{Z}^d$ and $A$ has integer entries, the vector $A^*(\mathbf{k}) \in \mathbb{Z}^d$. Therefore, if we do (4.1) again but now with $T(x)$ in place of $x$, $A^*(\mathbf{k})$ in place of $\mathbf{k}$, we still have that $\mathbf{k} \cdot T^2(x)$ is of the form $(A^*)^2(\mathbf{k}) \cdot x + q$ for the integer $q := A^*(\mathbf{k}) \cdot \lfloor A(T(x)) \rfloor \in \mathbb{Z}$. In general

$$\mathbf{k} \cdot T^n(x) = (A^*)^n(\mathbf{k}) \cdot x + q(x, \mathbf{k}, n)$$

for some $q(x, \mathbf{k}, n) \in \mathbb{Z}$ for any $n \in \mathbb{N}$ and $x \in \mathbb{T}^d$. This yields in particular that

$$\exp(2\pi i \mathbf{k} \cdot T^n(x)) = \exp(2\pi i (A^*)^n(\mathbf{k}) \cdot x).$$

Let $\sigma_d = \inf\{|Ax| : |x| = 1\}$ be the smallest singular value of $A$, which is the same as the smallest singular value of $A^*$. Thus we have for any $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$ that

$$|(A^*)^n y| \geqslant \sigma_d^n |y|. \tag{4.2}$$

Then we may estimate

$$r_N = \frac{1}{N^2} \sum_{m,n=1}^{N} \int \exp(-2\pi i \mathbf{k} \cdot (T^m(x) - T^n(x))) \, d\mu(x)$$

$$= \frac{1}{N^2} \sum_{m,n=1}^{N} \int \exp(-2\pi i ((A^*)^m(\mathbf{k}) - (A^*)^n(\mathbf{k})) \cdot x) \, d\mu(x)$$

$$\leqslant \frac{1}{N} + \frac{2}{N^2} \sum_{m=2}^{N} \sum_{n=1}^{m-1} |\widehat{\mu}((A^*)^m(\mathbf{k}) - (A^*)^n(\mathbf{k}))|.$$

By the mean value theorem, whenever $1 \leqslant n < m$, we can find $n \leqslant \varrho \leqslant m$ such that $\sigma_d^m - \sigma_d^n = [\sigma_d^\varrho \log \sigma_d](m-n) \geqslant c(m-n)$ for $c = \log \sigma_d > 0$ as $\sigma_d > 1$. Hence by the Fourier decay assumption and (4.2) we obtain

$$\sum_{m=2}^{N} \sum_{n=1}^{m-1} |\widehat{\mu}((A^*)^m(\mathbf{k}) - (A^*)^n(\mathbf{k}))| \lesssim \sum_{m=2}^{N} \sum_{n=1}^{m-1} |(A^*)^m(\mathbf{k}) - (A^*)^n(\mathbf{k})|^{-\alpha}$$

$$\leqslant \sum_{m=2}^{N} \sum_{n=1}^{m-1} \frac{1}{(\sigma_d^m |\mathbf{k}| - \sigma_d^n |\mathbf{k}|)^\alpha}$$

$$\leqslant \sum_{m=2}^{N} \sum_{n=1}^{m-1} \frac{1}{(m-n)^\alpha |c\mathbf{k}|^\alpha}$$

$$= O(N^{2-\gamma}),$$

where we used $|\mathbf{k}| \geqslant 1$ and $c > 0$. Since $\alpha > 0$ we have that

$$\sum_{N=1}^{\infty} \frac{r_N}{N} < \infty.$$

Thus by Lemma 4.1 there is a subsequence $N_j \to \infty$, $j \in \mathbb{N}$, such that

$$\sum_{j=1}^{\infty} r_{N_j} < \infty \quad \text{and} \quad \lim_{j \to \infty} \frac{N_{j+1}}{N_j} = 1.$$

In particular, the former condition yields that

$$\int \sum_{j=1}^{\infty} |S_{N_j}(x)|^2 \, d\mu(x) < \infty$$

and so $S_{N_j}(x) \to 0$ for $\mu$ almost every $x$ as $j \to \infty$. Now we just check that the latter condition actually yields $S_N(x) \to 0$ for $\mu$ almost every $x$ as $N \to \infty$, which is what we need. Fix $N \in \mathbb{N}$ and find $j \in \mathbb{N}$ such that $N_j \leqslant N \leqslant N_{j+1}$. This yields that

$$|NS_N(x) - N_j S_{N_j}(x)| = \left| \sum_{n=N_j+1}^{N} \exp(-2\pi i \mathbf{k} \cdot T^n(x)) \right| \leqslant N - N_j \leqslant N_{j+1} - N_j.$$

Hence
$$|S_N(x)| \leqslant |S_{N_j}(x)| + \frac{N_{j+1} - N_j}{N_j},$$
which converges to 0 as $j \to \infty$ at $\mu$ almost every $x$. $\qquad\qquad\qquad\square$

## 5. FURTHER RESEARCH

5.1. **Gaussian processes.** A promising further problem would be to try to extend this result for other Gaussian processes. The most probable next step would be to obtain bounds for the Fourier dimension of the graph of the *fractional Brownian motion* $B^H = B_t^H$ for a Hurst parameter $0 < H < 1$. This is a Gaussian process with mean 0 and the increments $B_t^H - B_s^H$ have the variance
$$\mathbb{E}|B_t^H - B_s^H|^2 = |t - s|^{2H}.$$
Note that in the case $H = 1/2$ the process $B_t^H$ is just the standard Wiener process $W_t$.

Kahane [19] already established an analogue of the bound on the Fourier transform for the images of Lebesgue under $B_t^H$, which means that we could bound the Fourier transform of the measure $\mu$ (Lebesgue measure lifted on the graph $G(B^H)$) for angles $\theta$ which are not too close to 0, $\pi$ or $2\pi$ with $|\widehat{\mu}(\xi)| = O(|\xi|^{-1/(2H)}\sqrt{\log|\xi|})$. However, the issue comes from the fact that Itō calculus is defined for Itō drift-diffusion processes for which the diffusion component only has the Wiener process. There are analogues of Itō calculus available for $B_t^H$ depending on the roughness of $B_t^H$, see for example [29] for an overview.

In the case $H \geqslant 1/2$ Itō's lemma for processes $X_t = b_t dt + \sigma_t dB_t^H$ has the form
$$f(X_T) - f(X_0) = \int_0^T b_t f'(X_t)\, dt + \int_0^T \frac{\sigma_t^2}{2} f''(X_t) t^{2H-1}\, dt + \int_0^T \sigma_t f'(X_t)\, dB_t^H,$$
where $\int X_t\, dB_t^H$ means a stochastic integral in the sense of pathwise Young integrals, see [38, 39]. However, if we recall the proof of Lemma 3.6 when we applied Itō's lemma, the problem now comes that even though $b_t$ and $\sigma_t$ are time-independent and $f'$ and $f$ are $-1$ or $i$ multiple of $f$, we will have the term $t^{2H-1}$ in the integral, which prevents us from having an equation
$$\int_0^T f(X_t)\, dt = c \int_0^T f(X_t)\, dB_t^H$$
for some constant $c \in \mathbb{C}$. Burkholder-Davis-Gundy type-inequalities have also been developed for fractional Brownian motions, see for example [28]. For small $H$ tools from rough path theory might be of help here. Moreover, the recently developed methods by Peres and Sousi [30] for the Hausdorff dimension of the graph of fractional Brownian motion with variable drift might be relevant.

5.2. **Other measures on the Brownian graph.** Theorem 1.3 concerns the push-forward of Lebesgue measure on $[0, 1]$ onto the graph $G(W)$ but it would be an interesting problem to see if one can have similar results for other possibly fractal measures on $[0, 1]$. A possible problem could be:

**Problem 5.1.** *Classify measures $\tau$ on $[0, 1]$ such that for some $0 < s \leqslant 1$ we have*
$$|\widehat{\tau}(\xi)| = O(|\xi|^{-s/2}), \quad |\xi| \to \infty,$$

*such that their lift $\mu_\tau$ onto the graph of $G(W)$ under $t \mapsto (t, W_t)$ has power decay with the exponent $-s/2$ for the Fourier transform.*

This is motivated by the fact that in Kahane's work [19] it is possible to transfer information on the Fourier decay (or Frostman properties) of $\tau$ onto the image measure. Thus for directions $\theta$ bounded away from 0 and $\pi$ we could still bound $\widehat{\mu_\tau}(\xi)$ using Kahane's work. The main problem in generalising our approach to fractal measures $\tau$ on $[0, 1]$ comes from the lack of Itō calculus. One would need to find a way to link the integrals $\int_0^T Z_t \, d\tau(t)$ to some form of stochastic integrals "$\int_0^T Z_t \, dW_t$" involving $\tau$.

5.3. **Prevalence.** Given that it is often difficult to study the graph of a specific continuous function, one is often interested in the generic case. Indeed, our results on the *almost sure* properties of the Brownian graph are of this nature. When one wants to study generic behaviour within the Banach space $(C[0, 1], \|\cdot\|_\infty)$, there are two main approaches: topological (using Baire category), or measure theoretic. The measure theoretic approach is often based on *prevalence*, introduced by Hunt, Sauer and Yorke [15], which plays the role of "Lebesgue almost all" in infinite dimensional Banach spaces in the absence of a useful analogue of Lebesgue measure. In our context the definition is as follows: a Borel set $\Lambda \subseteq C[0, 1]$ is *prevalent* if there exists a compactly supported Borel probability measure $\mathbb{P}$ on $C[0, 1]$ such that

$$\mathbb{P}\big(C[0, 1] \setminus (\Lambda + f)\big) = 0$$

for all $f \in C[0, 1]$.

The results of Mauldin and Williams [26] and Orponen with the authors [14] yield that the for a Baire generic $f \in C[0, 1]$ the graph has Hausdorff dimension 1 and Fourier dimension 0. For the prevalence case, Fraser and Hyde [10] proved that the graph of a prevalent $f \in C[0, 1]$ has Hausdorff dimension 2. This result was later generalised by Bayart and Heurteaux [2], who simplified the approach of Fraser and Hyde by using fractional Brownian motion. The main idea is that $\mathbb{P}$ is assumed to be the distribution of the fractional Brownian motion $t \mapsto B_t^H$ in $C[0, 1]$ for a given small $H$, and then one establishes that the graph $G(f + B^H)$ of the drifted process $f(t) + B_t^H$ has the same dimension as $G(B^H)$ almost surely for any fixed $f \in C[0, 1]$. By Adler's results [1], the Hausdorff dimension of $G(B^H)$ tends to 2 as $H \to 0$.

For Fourier dimension it could be possible to adapt the same approach. Since $G(W)$ has the maximal Fourier dimension 1 almost surely, it is likely that that the Fourier dimension of a prevalent $f \in C[0, 1]$ is 1. The problem reduces to proving that the graph measures $\mu_{f+W}$ on the drifted process $f(t) + W_t$ almost surely have the same Fourier decay as $\mu$ in Theorem 1.3 for any fixed $f \in C[0, 1]$. In this case the Fourier transform $\widehat{\mu}_{f+W}$ has the form $\widehat{\mu}_{f+W}(\xi) = \int_0^1 \exp(iY_t) \, dt$ for the process

$$Y_t := -2\pi u(t \cos \theta + f(t) \sin \theta + W_t \sin \theta),$$

which is no longer an Itō drift-diffusion process unless $f(t)$ is linear (compare this with (3.1)). Thus one would need to find ways to develop theory for such processes to adapt the approach presented here.

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